

Math Boot Camp: Div, Grad, Curl

You can skip this boot camp if you can answer the following question:

Example

Calculate the divergence of the vector field $\vec{A} = \hat{x} + \hat{y}$. Does the answer make sense?

Example

Calculate the curl of the vector field $\vec{A} = -y\hat{x} + x\hat{y}$. Do you expect the answer to be zero or nonzero?

Position

The position vector \vec{r} is defined as

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} \quad (1)$$

with norm

$$r = (x^2 + y^2 + z^2)^{1/2} \quad (2)$$

The corresponding unit vector (pointing radially outwards) equals

$$\hat{r} = \frac{\vec{r}}{r} = \frac{x\hat{x} + y\hat{y} + z\hat{z}}{(x^2 + y^2 + z^2)^{1/2}} \quad (3)$$

We define the infinitesimal displacement vector from the point (x, y, z) to the point $(x + dx, y + dy, z + dz)$ as

$$d\vec{s} = dx\hat{x} + dy\hat{y} + dz\hat{z} \quad (4)$$

(We could call this $d\vec{r}$, since that is what it is, but it is useful to reserve a special letter for infinitesimal displacements.)

Derivatives

Given a function of one variable $f[x]$, its derivative df tells us the rate of change of f ,

$$df = \left(\frac{df}{dx}\right) dx \quad (5)$$

In other words, if we change x by an amount dx , then f will change by an amount df , with the derivative being the proportionality factor. This has a nice geometrical interpretation as the slope of the function $f[x]$.

A function of 3 variables $T[x, y, z]$ requires us to generalize our definition of the derivative to

$$dT = \left(\frac{\partial T}{\partial x}\right) dx + \left(\frac{\partial T}{\partial y}\right) dy + \left(\frac{\partial T}{\partial z}\right) dz \quad (6)$$

where the notation $\frac{\partial T}{\partial x}$ is the partial derivative of T with respect to x (which means that y and z are treated as constants when taking this derivative (for example, if $T = 2x^2y$ then $\frac{\partial T}{\partial x} = 4xy$, $\frac{\partial T}{\partial y} = 2x^2$, and $\frac{\partial T}{\partial z} = 0$)). We could write this equation more compactly as

$$dT = \vec{\nabla} T \cdot d\vec{s} \quad (7)$$

where

$$\vec{\nabla} T \equiv \frac{\partial T}{\partial x} \hat{x} + \frac{\partial T}{\partial y} \hat{y} + \frac{\partial T}{\partial z} \hat{z} \quad (8)$$

is the gradient of T (i.e. the 3D analog of the derivative) and

$$d\vec{s} = dx\hat{x} + dy\hat{y} + dz\hat{z} \quad (9)$$

is the infinitesimal displacement vector from above. Using a property of the dot product,

$$dT = |\vec{\nabla} T| |d\vec{s}| \cos[\theta] \quad (10)$$

where θ is the angle between the vectors $\vec{\nabla} T$ and $d\vec{s}$. So if we fix the magnitude $|d\vec{s}|$ and rotate it around to cover all possible directions, then the maximum change in dT will occur when $\theta = 0$ (which implies that $d\vec{s}$ points in the same direction as $\vec{\nabla} T$). Therefore, a geometric interpretation of the gradient is that

$$\vec{\nabla} T \text{ points in the } \textit{direction of maximum increase of the function } T \quad (11)$$

and furthermore that

$$|\vec{\nabla} T| \text{ gives the slope of } T \text{ along this maximal direction} \quad (12)$$

Using the above expression, we also immediately see that $-\vec{\nabla} T$ is the direction of maximum *decrease* of the function T , and that any direction perpendicular to $\vec{\nabla} T$ will result in no change to T (i.e. this plane will have a constant T value). Of course, all of these statements only hold for an *infinitesimal* displacement $d\vec{s}$.

We can write

$$\begin{aligned} \vec{\nabla} T &= \frac{\partial T}{\partial x} \hat{x} + \frac{\partial T}{\partial y} \hat{y} + \frac{\partial T}{\partial z} \hat{z} \\ &= \left(\hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right) T \end{aligned} \quad (13)$$

where

$$\vec{\nabla} = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \quad (14)$$

is the del operator. Note that the derivatives in $\vec{\nabla}$ each act separately on T . Also, because the unit vectors never change magnitude or direction (i.e. $\frac{\partial \hat{x}}{\partial x} = 0$), you typically see $\vec{\nabla}$ written as $\vec{\nabla} = \frac{\partial}{\partial x} \hat{x} + \frac{\partial}{\partial y} \hat{y} + \frac{\partial}{\partial z} \hat{z}$, which is how I will write it throughout the rest of the notes.

Note that $\vec{\nabla}$ is not a normal vector, but instead an operator that acts upon a scalar to produce a vector (so $\vec{\nabla} T$ does not mean ordinary multiplication). In fact, we know of three ways that vectors can be multiplied:

- Multiply vector \vec{A} by a scalar c : $c\vec{A}$
- Take the dot product of two vectors \vec{A} and \vec{B} : $\vec{A} \cdot \vec{B}$
- Take the cross product of two vectors \vec{A} and \vec{B} : $\vec{A} \times \vec{B}$

Correspondingly, the $\vec{\nabla}$ operator can act in three ways:

- On a scalar function T : $\vec{\nabla} T$ (the gradient of T)
- On a vector \vec{A} via the dot product: $\vec{\nabla} \cdot \vec{A}$ (the divergence of \vec{A})
- On a vector \vec{A} via the cross product: $\vec{\nabla} \times \vec{A}$ (the curl of \vec{A})

We already discussed the gradient above, so we now focus on the last two operations.

Divergence

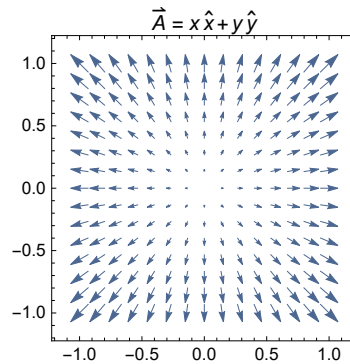
The divergence of a vector $\vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$ equals

$$\begin{aligned} \vec{\nabla} \cdot \vec{A} &= \left(\frac{\partial}{\partial x} \hat{x} + \frac{\partial}{\partial y} \hat{y} + \frac{\partial}{\partial z} \hat{z} \right) \cdot (A_x \hat{x} + A_y \hat{y} + A_z \hat{z}) \\ &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \end{aligned} \quad (15)$$

Note that the divergence of a vector is a scalar (and there is no such thing as the divergence of a scalar)! Geometrically, the divergence of a vector \vec{A} represents how much \vec{A} spreads out from any point. More precisely, if you treat \vec{A} as describing the flow of a fluid and you put an infinitesimally small volume at a point \vec{r} , then the volume will empty out in time if the divergence is positive, fill up with time if the divergence is negative, and hold the same

amount of fluid if the divergence equals zero. In other words, the divergence measures the "amount of arrow" leaving an infinitesimal volume at each point in space, with larger arrows counting more than smaller arrows.

An Important Point: In this course, we will often work with vectors which are themselves functions of position. For example, $\vec{A} = x \hat{x} + y \hat{y}$ is a different vector at each point and space (and it has a positive divergence equal to 2 at each point in space). We can sample different points in space to get a feel for what this looks like, as shown below. These "vectors" are actually called [vector fields](#).

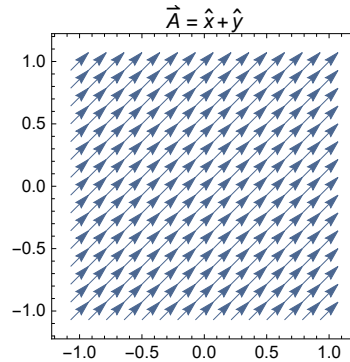


Example

Calculate the divergence of the vector field $\vec{A} = \hat{x} + \hat{y}$. Does the answer make sense?

Solution

Since this is a constant vector field, the amount of arrows entering any infinitesimal volume equals the number of arrows exiting that volume.



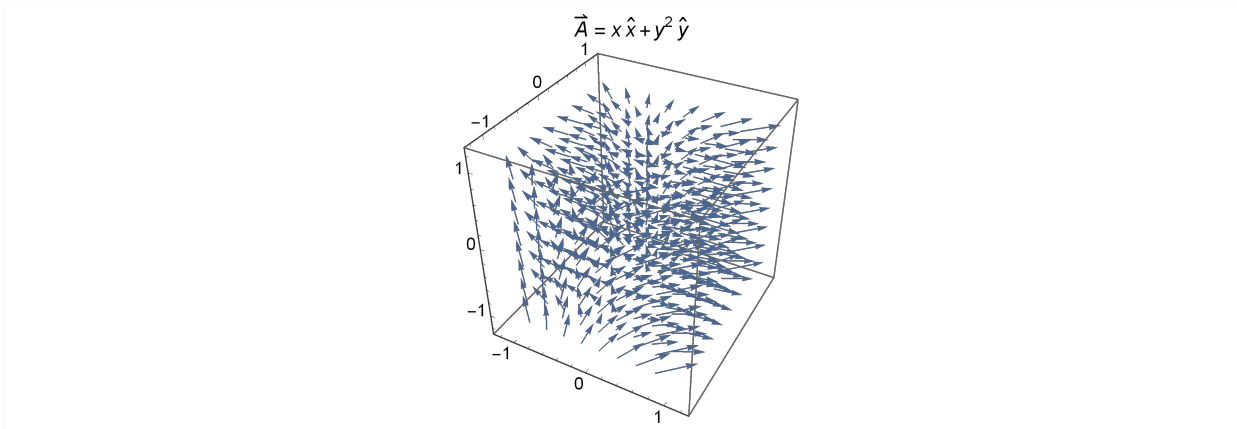
And indeed, $\vec{\nabla} \cdot \vec{A} = \frac{\partial}{\partial x}[1] + \frac{\partial}{\partial y}[1] = 0$, as expected. \square

Example

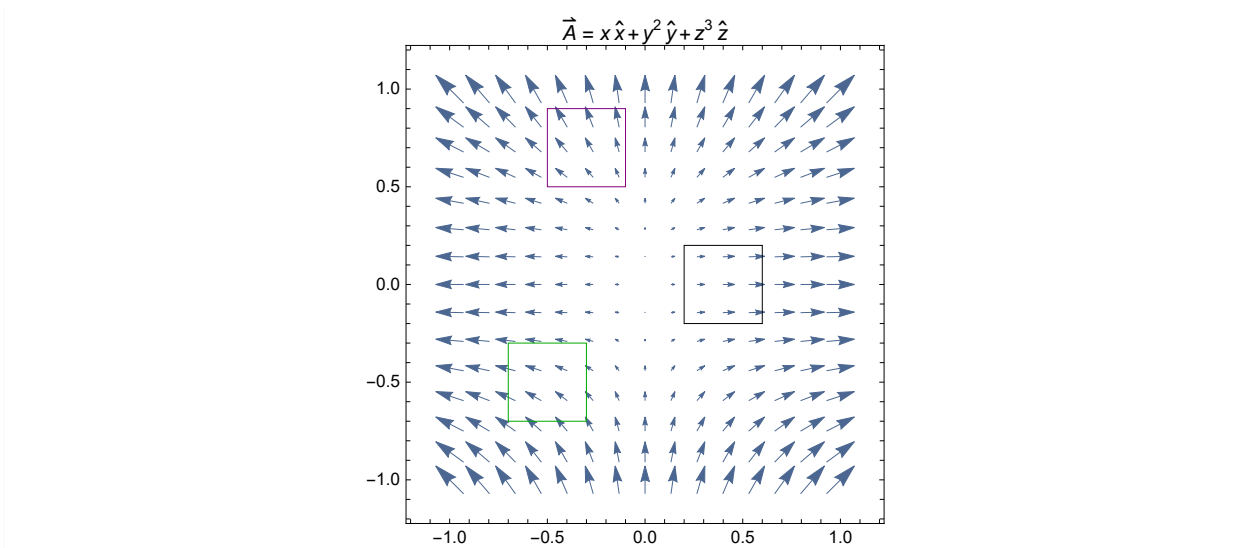
Calculate the divergence of the vector field $\vec{A} = x \hat{x} + y^2 \hat{y}$.

Solution

Plugging into our equation, $\vec{\nabla} \cdot \vec{A} = \frac{\partial x}{\partial x} + \frac{\partial y^2}{\partial y} = 1 + 2y$. But let's visually check this by plotting the 3D vector field.



This plot - while correct - is extremely cluttered. Because \vec{A} does not depend upon z , we can simply take a cut of this vector field at $z = 0$.



Consider the point $(0.4, 0)$ shown by the black square above. There are no vectors going in or out of it in the y -direction, but there are definitely larger vectors coming out of the square than going into it. This signifies that there will be positive divergence, in line with the result $(1 + 2y)_{x=0.4,y=0} = 1$ found above. Similarly, the green square around $(-0.5, -0.5)$ has larger vectors entering it than leaving it on the top and bottom and larger vectors leaving it than entering it on its sides. These two effects seem rather equal, and indeed we find $(1 + 2y)_{x=-0.5,y=-0.5} = 0$ from above. Lastly, the purple square centered at $(-0.3, 0.7)$ has divergence $(1 + 2y)_{x=-0.3,y=0.7} = 2.4$ which agrees with the fact that larger arrows are leaving the square than entering it. This shows the geometrical description of the divergence, but be warned: you must use an *infinitesimal squares* for this to be correct (I only showed a large square here for convenience!) \square

Example

Compute the divergence of the vector field $\vec{A} = \frac{\hat{r}}{r^2}$.

Solution

Recall that $\vec{r} = \langle x, y, z \rangle$, so that $r^2 = x^2 + y^2 + z^2$ and $\hat{r} = \frac{\langle x,y,z \rangle}{(x^2+y^2+z^2)^{1/2}}$. So in Cartesian coordinates,

$$\vec{A} = \left\langle \frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}} \right\rangle \quad (16)$$

Therefore,

$$\vec{\nabla} \cdot \vec{A} = \frac{\partial}{\partial x} \left[\frac{x}{(x^2+y^2+z^2)^{3/2}} \right] + \frac{\partial}{\partial y} \left[\frac{y}{(x^2+y^2+z^2)^{3/2}} \right] + \frac{\partial}{\partial z} \left[\frac{z}{(x^2+y^2+z^2)^{3/2}} \right] \quad (17)$$

Computing the first of these terms,

$$\begin{aligned} \frac{\partial}{\partial x} \left[\frac{x}{(x^2+y^2+z^2)^{3/2}} \right] &= \frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3x^2}{(x^2+y^2+z^2)^{5/2}} \\ &= \frac{-2x^2+y^2+z^2}{(x^2+y^2+z^2)^{5/2}} \end{aligned} \quad (18)$$

By symmetry (i.e. by redefining $x \rightarrow y$, $y \rightarrow z$, and $z \rightarrow x$), the other two terms must be

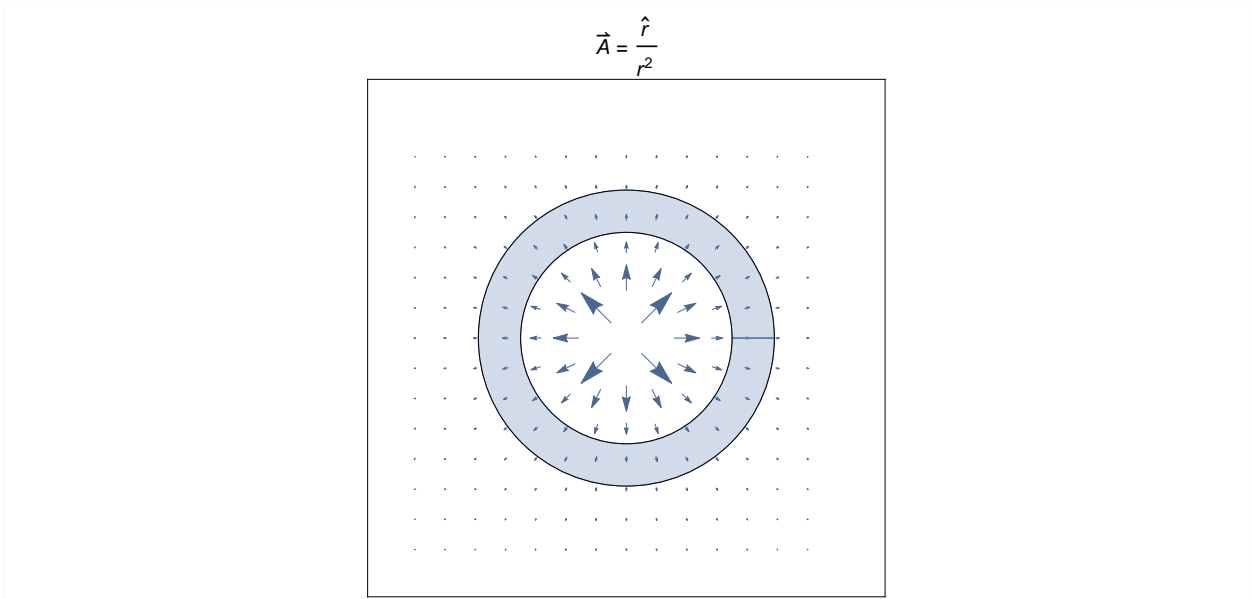
$$\frac{\partial}{\partial y} \left[\frac{y}{(x^2+y^2+z^2)^{3/2}} \right] = \frac{x^2-2y^2+z^2}{(x^2+y^2+z^2)^{5/2}} \quad (19)$$

$$\frac{\partial}{\partial z} \left[\frac{z}{(x^2+y^2+z^2)^{3/2}} \right] = \frac{x^2+y^2-2z^2}{(x^2+y^2+z^2)^{5/2}} \quad (20)$$

and therefore

$$\vec{\nabla} \cdot \vec{A} = 0 \quad (21)$$

That is pretty shocking, for if we draw this vector field (taking a cut at $z = 0$), what do we expect for its divergence to be?



But upon closer inspection, maybe this isn't so crazy. Let's consider the area between a circle of radius r_1 and r_2 where $r_1 < r_2$. Note that the divergence represents the amount of the arrows leaving an infinitesimal volume at each point in space, and not on a finitely large circle. However in the special case where the divergence equals 0 everywhere, that means that at each infinitesimal volume in space, the amount of arrows leaving that point equal the amount of arrows entering it, so *in this case only* the amount of arrows entering a finite volume should equal the amount of arrows leaving a finite volume. Let's check this for the donut shown above.

The arrows enter the donut all along its inner perimeter, which has length $2\pi r_1$, and these arrows have magnitude $\frac{1}{r_1^2}$. Therefore, the cumulative effect of all these arrows on the inner perimeter equals $(2\pi r_1) \left(\frac{1}{r_1^2} \right) = \frac{2\pi}{r_1}$. Similarly, the arrows leaving the donut occur at the perimeter $2\pi r_2$ with magnitude $\frac{1}{r_2^2}$. Therefore, the weight of these arrows equals $(2\pi r_2) \left(\frac{1}{r_2^2} \right) = \frac{2\pi}{r_2}$. So we expect that the divergence, which equals the weight of the arrows leaving the

figure minus the weight of the arrows entering the figure, should equal $\frac{2\pi}{r_2} - \frac{2\pi}{r_1}$. Since $r_1 < r_2$, this is a negative number, and certainly not zero. So what went wrong?

The problem is that we are working in 2D, rather than 3D! Indeed, in 2D the divergence is negative (in fact, it is $-\frac{1}{r^2}$), but in 3D we have to take into account the surface area of the inner and outer *spheres*. Therefore, the weight of the arrows entering the inner sphere equals $(4\pi r_1^2)\left(\frac{1}{r_1^2}\right) = 4\pi$ while the weight of the arrows leaving the outer sphere equals $(4\pi r_2^2)\left(\frac{1}{r_2^2}\right) = 4\pi$, and these two contributions do indeed cancel to leave a net 0 divergence!

Great, but we still have one more mystery to solve. While it may seem reasonable that the divergence of $\vec{A} = \frac{\hat{r}}{r^2}$ is 0 at every point, there is one point at which this cannot be so: the origin. For at the origin, there are vectors emanating outwards in every direction, so putting an infinitesimal sphere at the origin must surely yield a positive divergence. In fact, as it turns out, the divergence at the origin is infinitely large, and later in the course we will quantify this infinity explicitly to determine the electric potential from a point charge.

Curl

We now turn to the final operation you can do with the $\vec{\nabla}$ operator, namely, taking the *curl* of a vector $\vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$, which equals

$$\begin{aligned} \vec{\nabla} \times \vec{A} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \\ &= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \hat{x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \hat{z} \end{aligned} \quad (22)$$

Note that the curl of a vector yields *another vector*. You cannot take the curl of a scalar, or a 1D or 2D vector. Therefore, when taking the curl of a vector such as $\vec{A} = \hat{x} + \hat{y}$ we will always assume it is a 3D vector with zero for its \hat{z} component.

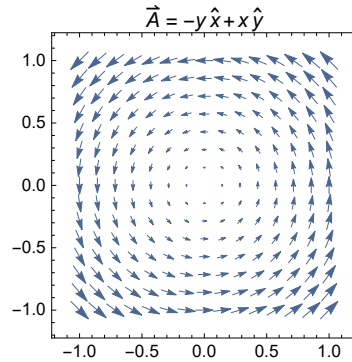
The curl measures how much a vector field "curves around" a point. If an object in your vector field would rotate (ignore its translation and just focus on rotation) then that point has a non-zero curl. For example, considering the motion of water in a pond, a whirlpool would be a region of large curl with the direction of the curl given by the right-hand rule (i.e. when you wrap your fingers in the direction of the rotation, your thumb points in the direction of the curl).

Example

Calculate the curl of the vector field $\vec{A} = -y \hat{x} + x \hat{y}$. Do you expect the answer to be zero or nonzero?

Solution

We expect from the shape of the vector field that there will be a curl in the $+\hat{z}$ direction.



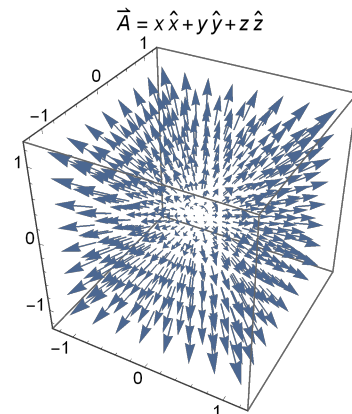
And indeed, $\vec{\nabla} \times \vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \hat{x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \hat{z} = 2 \hat{z}$. This may perhaps seem like an obvious statement about the origin, but the result means more than that, since $\vec{\nabla} \times \vec{A} = 2 \hat{z}$ implies that placing a small object at *any* point will result in that object rotating at the same rate (although for any point that isn't the origin the object will also translate). \square

Example

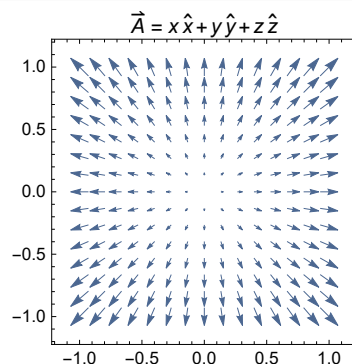
Calculate the curl of the vector field $\vec{A} = x \hat{x} + y \hat{y} + z \hat{z}$.

Solution

This vector field points radially outward. What do we expect the curl to be?



From our formula, $\vec{\nabla} \times \vec{A} = 0$. Does this make sense? Looking at the plane $z = 0$,



we see that although any small object would certainly be pushed radially outwards, it will *not rotate*. For example, placing a sphere - which in the $z = 0$ plane is a circle - at the point $(0, 1)$ shows that the rotation on the points in the

left side of the sphere will be perfectly balanced by the rotation on the points in the right side of the sphere. Because the sphere will not rotate, its curl must be zero. This applies to a circle at any other point by the symmetry of the vector field. \square

Mathematica Initialization